2.4

#19 $\overset{M}{\left(2x + \frac{y}{1+x^2y^2}\right)} dx + \overset{N}{\left(\frac{x}{1+x^2y^2} - 2y\right)} dx = 0$

exact? $\frac{\partial M}{\partial y} = \frac{(1+x^2y^2)[1] - y[2x^2y]}{(1+x^2y^2)^2}$

$$= \frac{1+x^2y^2-2x^2y^2}{(1+x^2y^2)^2} = \frac{1-x^2y^2}{(1+x^2y^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{(1+x^2y^2)[1] - x[2xy^2]}{(1+x^2y^2)^2} = \frac{1-x^2y^2}{(1+x^2y^2)^2}$$ ∴ EXACT!

So begin with M

$$F(x,y) = \int \left(2x + \frac{y}{1+x^2y^2}\right) dx = x^2 + \tan^{-1}(xy) + \phi(y)$$

* Note: $y \int \frac{1}{1+x^2y^2} dx = y \int \frac{1}{1+(xy)^2} dx$

let $u = xy$
$du = y\,dx$
$\frac{1}{y} du = dx$

∴ $\int \frac{1}{1+u^2} du = \underline{\tan^{-1}(u)}$

So $F(x,y) = x^2 + \tan^{-1}(xy) + \phi(y)$

Now we know $\frac{\partial F}{\partial y} = N$, so

$$\frac{\partial F}{\partial y} = \frac{1}{1+(xy)^2}[x] + \phi'(y) = \frac{x}{1+x^2y^2} - 2y$$

$\therefore \phi'(y) = -2y$

$\phi(y) = -y^2$

Finally, $\underline{F(x,y) = x^2 + \tan^{-1}(xy) - y^2 = C}$

2.4

#16 $\overbrace{(ye^{xy} - y^{-1})}^{M}dx + \overbrace{(xe^{xy} + xy^{-2})}^{N}dy = 0$

exact? $\frac{\partial M}{\partial y} = yxe^{xy} + e^{xy} + \frac{1}{y^2}$

$\frac{\partial N}{\partial x} = xye^{xy} + e^{xy} + y^{-2}$ $\therefore$ exact !

Select $M \Rightarrow F(x,y) = \int (ye^{xy} - y^{-1})dx = y\left[\frac{1}{y}e^{xy}\right] - xy^{-1} + \phi(y)$ *

$F(x,y) = e^{xy} - xy^{-1} + \phi(y)$

and we know $\frac{\partial F}{\partial y} = N$ then

$\frac{\partial F}{\partial x} = xe^{xy} + xy^{-2} + \phi'(y) = xe^{xy} + xy^{-2}$

So $\phi'(y) = 0 \Rightarrow \phi(y) = C_1$, $C_1 \in \mathbb{R}$

$\therefore$ $F(x,y) = e^{xy} - \frac{x}{y} = C_2$, where $\underline{C_2 = C - C_1}$

2.4

#13 $\overset{M}{(t y^{-1})}\, dy + \overset{N}{(1+\ln y)}\, dt = 0$

exact? $\dfrac{\partial M}{\partial t} = y^{-1} = \dfrac{\partial N}{\partial y}$ so exact!

Select $M \Rightarrow F(y,t) = t\int y^{-1} dy = t\ln y + \phi(t)$

and $\dfrac{\partial F}{\partial t} = N \Rightarrow \dfrac{\partial F}{\partial t} = \ln y + \phi'(t) = 1 + \ln y$

$\therefore \phi'(t) = 1 \Rightarrow \phi(t) = t$

Giving $\underline{F(y,t) = t\ln y + t = C}$

#25 $(y^2 \sin x)dx + (x^{-1} - y x^{-1})dy = 0, \quad y(\pi) = 1$

exact? $\dfrac{\partial M}{\partial y} = 2y\sin x \neq \dfrac{\partial N}{\partial x} = -x^{-2} + y x^{-2} = \dfrac{y-1}{x^2}$

So Not exact! So rewrite

$(y^2 \sin x)dx - \left(\dfrac{y-1}{x}\right)dy = 0 \Rightarrow (y^2 \sin x)dx = \dfrac{y-1}{x}dy$

OR $x\sin x\, dx = \dfrac{y-1}{y^2}dy = \left(\dfrac{1}{y} - \dfrac{1}{y^2}\right)dy = (y^{-1} - y^{-2})dy$

Separable $\int x\sin x\, dx = \int (y^{-1} - y^{-2})dy$

$-x\cos x + \sin x + C_1 = \ln|y| + \dfrac{1}{y} + C_2$

$\dfrac{1}{y} + \ln|y| = -x\cos x + \sin x + C_3, \quad C_3 = C_1 - C_2$

$\therefore 1 + 0 = -\pi\cos(\pi) + \sin\pi + C_3 \Rightarrow 1 = \pi + C_3$

$1 - \pi = C_3.$